L5. Production Theory

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Literature

- MWG (1995), Chapter 5
- Kreps (1990), Chapter 7; Varian (1992), Chapters 1-5

Introduction

This chapter is often called "theory of the firm", but this is not quite appropriate. The "neoclassical theory of the firm" sees the firm merely as a production function, a black box that transforms inputs into outputs. Incentives, ownership rights, boundaries of the firm, and organizational issues are all left out of the picture.

Topics of this lecture:

- Description of the production possibilities
- Profit maximization
- Cost functions and factor demand correspondences
- Aggregation and efficient production
- Objectives of the firm

Production Sets

In order to describe the production possibilities of the firm, we use the following concepts:

- **L commodities**, indexed by $l \in \{1, \ldots, L\}$
- **Production plan** (netput vector): $y = (y_1, \ldots, y_L) \in \mathbb{R}^L, y_l < 0$ (net input), $y_l > 0$ (net output).
- **Production set:** $Y \subset \mathbb{R}^L$ contains all feasible production plans. Any $y \in Y$ is feasible, any $y \notin Y$ is not.
- **Transformation function:** $F(y)$, is defined by $Y = \left\{y \in \mathbb{R}^L \mid F(y) \leq 0\right\}$ and $F(y) = 0$ if and only if y is on the boundary of the production set.
- **Transformation frontier:** $\left\{y \in \mathbb{R}^L \mid F(y) = 0\right\}$.
- **Production function:** Suppose that each good is either an input or an output. Let $q = (q_1, \ldots, q_M)$ be an output vector and $z = (z_1, \ldots, z_{L-M})$ be an input vector. Then we can describe the production set by a production function $q = f(z)$. This is usually done in the special but important case of a single output good.

- **Marginal rate of transformation:** Suppose that $F(\cdot)$ is differentiable and that $\bar{y}$ satisfies $F(\bar{y}) = 0$. Then we have

$$\frac{\partial F(\bar{y})}{\partial y_k} dy_k + \frac{\partial F(\bar{y})}{\partial y_l} dy_l = 0$$

or

$$MRT_{lk}(\bar{y}) = -\frac{dy_k}{dy_l} = \frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k}$$

  – Interpretation?

- **Marginal Rate of Technical Substitution:**

$$MRTS_{lk} = \frac{\partial f(\bar{z})/\partial z_l}{\partial f(\bar{z})/\partial z_k}$$

  – Interpretation?

Commonly assumed properties of production sets:

Not all of the following assumptions do always make sense. Some of them are mutually exclusive.

1. Y is nonempty.
2. Y is closed, i.e., it contains its boundary.
3. No free lunch: if $y \geq 0$ and $y \in Y$, then $y = 0$. Put differently, $Y \cap \mathbb{R}^L_+ \subset \{0\}$.
4. Possibility of inaction: $0 \in Y$. If inaction is not possible, then there are sunk costs.
5. Free disposal: if $y \in Y$ and $y' \leq y$, then $y' \in Y$.
6. Irreversibility: if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.
7. Non-increasing returns to scale: for any $y \in Y$ and $\alpha \in [0, 1]$, we have $\alpha y \in Y$. That is, any feasible production plan can be scaled down.
8. Non-decreasing returns to scale: for any $y \in Y$ and any $\alpha \geq 1$, we have $\alpha y \in Y$. That is, any feasible production plan can be scaled up.
9. Constant returns to scale: for any $y \in Y$ and any $\alpha \geq 0$, we have $\alpha y \in Y$. That is, any feasible production plan can be scaled up and scaled down. Geometrically, Y is a cone.
10. Additivity: if $y \in Y$ and $y' \in Y$, then $y + y' \in Y$. Additivity is different from non-decreasing returns to scale, because it can also be applied to integer problems.
11. Convexity: if $y, y' \in Y$ and $\alpha \in [0, 1]$, then

$$\alpha y + (1 - \alpha)y' \in Y.$$

Profit Maximization

Assumptions:

1. **Price-taking behavior:** The vector $p = (p_1, \ldots, p_L)$ of prices is exogenously given and independent of the production plans of the firms. Discussion.

2. **Profit maximization:** The firm's only objective is to maximize profits. Discussion.

3. **Technical assumptions:** The production set satisfies non-emptiness, closeness, and free disposal. Discussion.

The profit maximization problem (PMP) is very similar to the utility maximization problem that we have already seen, but it has more structure and yields much more powerful empirical predictions.

$$\max_{y} p \cdot y \quad \text{s.t. } y \in Y,$$

or, alternatively,

$$\max_{y} p \cdot y \quad \text{s.t. } F(y) \leq 0$$

- Lagrange
- Geometry of profit maximization.

Let $y(p)$ denote the solution to this problem. We call $y(p)$ the **supply correspondence**.

Note:

- $y(p)$ need not be unique.
- It may be that there exists no solution to PMP. This is frequently the case if we have non-decreasing returns to scale. Why?
- Recall that if $y_k(p) > 0$, then $y_k(p)$ is the firm's supply of output k. If $y_l(p) < 0$, then $y_l(p)$ is the firm's demand for input l.

We can now define the **profit function** $\pi(p)$ by

$$\pi(p) = \max\{p \cdot y \mid y \in Y\}$$

If $y(p)$ is unique, then

$$\pi(p) = p \cdot y(p).$$

Note that the profit function is very similar to the indirect utility function. There is one important difference, however. The indirect utility function depends on prices and wealth, while the profit function depends only on prices! There is no budget constraint in production theory and, hence, there are **no wealth effects**.

Proposition 5.1 Suppose that $y(p)$ is the solution of the PMP given production set Y and $\pi(p)$ is the corresponding profit function. Then:

(a) $y(p)$ is homogeneous of degree 0.

(b) $\pi(p)$ is homogeneous of degree 1.

(c) $\pi(p)$ is convex.

(d) If Y is convex, then $y(p)$ is a convex set for all p. If Y is strictly convex, then $y(p)$ is single-valued.

(e) **Hotelling's Lemma:** If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.

(f) If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $D_p y(\bar{p}) = D_p^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Remarks:

- The proofs parallel those of Section 3.
- Hotelling's Lemma is very useful. If we know the profit function of a firm, we can directly compute it's supply function.
- The positive semidefiniteness of the matrix $Dy(p)$ is the **law of supply**: quantities respond in the same direction as price changes.
- The law of supply can also be expressed as

$$(p - p') \cdot (y - y') \geq 0$$

  It follows immediately from a revealed preference argument:

$$(p - p') \cdot (y - y') = (p \cdot y - p \cdot y') + (p' \cdot y' - p' \cdot y) \geq 0$$

- Note that the law of supply holds for any price change. In contrast to demand theory, there are no budget constraints and no wealth effects. There are only substitution effects in production theory!

Cost Minimization

Cost minimization is a necessary (but not sufficient) condition for profit maximization. It is useful to study this problem separately for several reasons:

1. When a firm is not a price taker on the output market, we can no longer use the profit function, but the results from cost minimization continue to hold as long as input prices are given.
2. If there are non-decreasing returns to scale, PMP does not have a solution, but the results from cost minimization can still be applied.
3. The cost minimization problem is useful to characterize the factor demand of the firm.

We only consider the single output case. In this case let z be the vector of inputs, $f(z)$ the production function, q the output, and w the vector of input prices.

Cost Minimization Problem (CMP):

$$\min_{z \geq 0} w \cdot z \quad \text{s.t. } f(z) \geq q.$$

Let $z(w, q)$ be a solution to this problem. It is called the **conditional factor demand correspondence**. For any $z^* \in z(w, q)$ the following first order conditions must hold:

$$w_l \geq \lambda \frac{\partial f(z^*)}{\partial z_l}, \text{ with equality if } z_l^* > 0.$$

- Illustrate graphically.
- Show that $(z_l, z_k) \gg (0,0)$ implies $MRTS_{lk} = w_l/w_k$.
- Interpretation of the Lagrange multiplier λ.

We can now define the **cost function** as the optimized value of CMP. If $z(p,w)$ is single valued we can write:

$$c(w,q) = w \cdot z^*(w,q)$$

Proposition 5.2 Suppose that $c(w, q)$ is the cost function of production function $f(\cdot)$ and that $z(w, q)$ is the corresponding conditional factor demand correspondence. Then

(a) $z(\cdot)$ is homogeneous of degree zero in w.

(b) $c(\cdot)$ is homogeneous of degree one in w and non-decreasing in q.

(c) $c(\cdot)$ is a concave function of w.

(d) If the set $\{z \geq 0 \mid f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. If $\{z \geq 0 \mid f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is single-valued.

(e) Shephard's Lemma: if $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(f) If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$

All of these results follow immediately from the analysis of the expenditure minimization problem in Chapter 3. Simply replace $u(\cdot)$ by $f(\cdot)$, u by q, and x by z (i.e., interpret the utility function as a production function).

Aggregation and Efficient Production

The aggregate supply correspondence is simply the sum of the individual supply correspondences. Assuming that $y_j(p)$ is single-valued for all firms $j \in \{1, \ldots, J\}$ we can write

$$y(p) = \sum_{j=1}^{J} y_j(p)$$

We know that for each firm $D_p y_j(p)$ is a symmetric, positive semidefinite matrix. Hence, $D_p y(p)$ must also be symmetric and positive semidefinite, which implies that **the law of supply holds in the aggregate.**

This can also be shown directly: For all $j \in \{1, \ldots, J\}$ we have

$$(p - p') \cdot [y_j(p) - y_j(p')] \geq 0$$

Summing up over j yields

$$(p - p') \cdot [y(p) - y(p')] \geq 0.$$

In contrast to demand theory, there are no wealth effects, which simplifies aggregation dramatically.

We can also describe the production side of the economy through a **representative producer**. Define the **aggregate production set** by

$$Y = Y_1 + \ldots + Y_J$$

Let $\pi^*(p)$ and $y^*(p)$ be the corresponding profit function and the supply correspondence. The following proposition shows that the aggregate profit obtained by each individual firm that maximizes profits separately is the same as the maximum profit that can be obtained if all firms merged and produced as a single firm.

Proposition 5.3 For all $p \gg 0$, we have:
(a) $\pi^*(p) = \Sigma_j \pi_j(p)$
(b) $y^*(p) = \Sigma_j y_j(p)$

Proof: We only prove (a).

The result tells us that the allocation of production across firms is cost minimizing. Why?

We have seen already that the profit maximizing production plans maximize aggregate profits and minimize aggregate costs. This suggests that the profit maximizing production plan is efficient in the following sense:

Definition 5.1 A production plan $y \in Y$ is efficient it there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Roughly speaking, a production plan is efficient, if there is no other feasible production plan that generates more output without using more inputs.

Proposition 5.4 If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

Proof:

Remarks:

- The proposition is a variant of the First Fundamental Theorem of Welfare.
- The result shows that profit maximization under price-taking behavior and efficiency are closely related.
- Note that the result does not require the production set to be convex.
- But, the result implicitly makes another important assumption. Which one?

The converse of this proposition would say that every efficient production plan is profit maximizing for some price vector p. However, this is only true if we assume that the production set is convex.

Proposition 5.5 Suppose that Y is convex. Then every efficient production plan $y \in Y$ is a profit-maximizing production plan for some price vector $p \geq 0$.

Remarks:

- This result is a variant of the Second Fundamental Theorem of Welfare.
- The proof is an application of the Separating Hyperplane Theorem.
- The result says that we can decentralize efficient production to profit maximizing firms by using the appropriate price vector.

Objectives of the Firm

It is natural to assume that a consumer wants to maximize his utility. It is much less natural to assume that a firm wants to maximize its profit:

- Why don't we assume that the firm wants to maximize a utility function which may include profits but which may also include sales revenues, market share, size of the labor force, etc. as additional arguments?
- If the firm has several owners, they may have different objectives. How do they resolve their potential conflicts of interest?

Suppose that the firm is owned by consumers. Each consumer $i \in 1, \ldots, I$, is entitled to share $\theta_i \geq 0$ of the firm's profits, with $\Sigma_{i=1}^{I} \theta_i = 1$. Some θ_i may be zero. Each owner wants to maximize his utility function:

$$\max_{x^i \geq 0} u^i\left(x^i\right)$$

subject to

$$p \cdot x^i \leq w_i + \theta_i [p \cdot y]$$

where w_i is consumer i 's non-profit wealth. Higher profits increase each consumer's budget. Hence, for any fixed price vector p, the consumer-owners unanimously prefer higher profits to lower profits, i.e., they agree on a policy of profit maximization! Hence, there are no conflicts of interests.

Remarks:

This argument rests on several important assumptions.

1. Prices are given. If prices are affected by the production of the firm, then the objectives of an owner may depend on his tastes as a consumer.
2. No uncertainty. If profits are uncertain and owners have different risk attitudes, they may have different objectives.
3. There are no incentive problems within the firm. If managers have their own objectives, it may be difficult for the owners to implement their preferred policy.