L8. Strategic Interactions

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Literature

- MWG, Chapter 7-9

Introduction

- So far, the decision maker's well-being depended only on the choices she made. She takes the market environment as given and acts optimally.
- In this lecture, we consider the presence of **strategic interdependence** – the actions that are best for the decision maker depend on actions that other individuals have already taken, or she expects them to be taking at the same time, or even in future actions that they may take, or decide not to take, as a result of current actions.

The tool that we use for analyzing settings with strategic interdependence is **non-cooperative game theory.**

Big Picture

Economic problems can generally be characterized as multi-agent interactions.

- Depending on the degree to which strategic interactions are present:
- e.g. firms as decision makers: monopoly $\rightarrow$ oligopoly $\rightarrow$ perfect competition

At the two extremes, the nature of strategic interaction is minimum enough that we do not need to make a formal use of game theory.

- In most of the course, we talked about agent behavior under perfect competition: atomic agent.

- The monopoly case is, by definition, also only one agent matters (only differ from PC in GE).

- Now we study the middle ground.

Outline

- Basic elements
- Simultaneous-move games
- Dynamic games

Basic Elements

Basic Elements of Non-cooperative Games

A game is a formal representation of a situation in which a number of individuals interact in a setting of *strategic independence*.

Four elements:

1. The players: who is involved?
2. The rules: who moves when? What do they know when they move? What can they do?
3. The outcomes: for *each possible set of actions* by the players, what is the outcome of the game?
4. The payoffs: what are the players' preferences (i.e., utility functions) over the possible outcomes?

Basic Elements of Non-cooperative Games

Four elements:

2. The rules:

   a Who moves when?
   - Simultaneous vs. Dynamic Games

   b What do they know when they move?
   - Perfect vs. Imperfect Information, Complete vs. Incomplete Information

   c What can they do?
   - Strategies : Pure vs. Random

Who moves when? - Simultaneous Moves

Normal Form Representation of a Game

		Player 2	
		Head	Tail
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Figure: Normal form for Matching Pennies Version A

Who moves when? - Dynamic Moves

Extensive Form Representation of a Game (Game Tree)

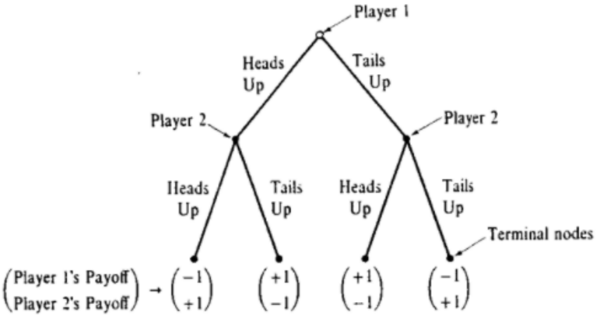


Figure 7.C.1
Extensive form for Matching Pennies Version B.

Some terms: initial decision node, player X's decision node, terminal nodes, root.

What do they know when they move?

Information Set

If an agent can observe all her rival's previous moves, they are games of perfect information.

The concept of information set allows us to accommodate the possibility that this is not so.

- a player can't distinguish which decision node she is actually at, among all nodes in the same information set.

Definition 8.1 A game is one of perfect information if each set contains a single decision node. Otherwise it is a game of imperfect information.

What do they know when they move?

Games with Imperfect Information

An Example

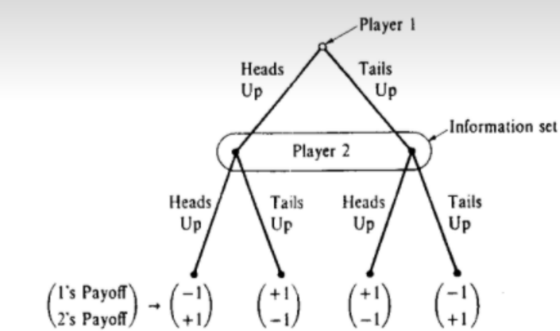


Figure 7.C.3
Extensive form for Matching Pennies Version C.

Remark: simultaneous games are trivially of imperfect information. Hence you only come across information problems in dynamic games.

What do they know when they move?

Games with Imperfect Information

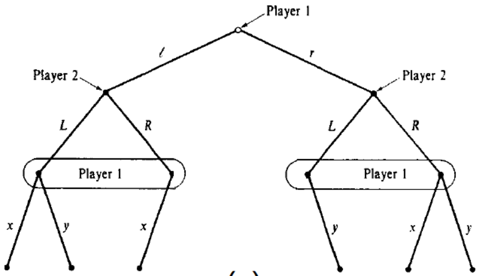
Two natural restrictions on information sets:

- 1: at every node within a given information set, a player must have the same set of possible actions – otherwise the player can "back out" his position!
- 2 (perfect recall): a player does not forget what she once knew, including her own actions.

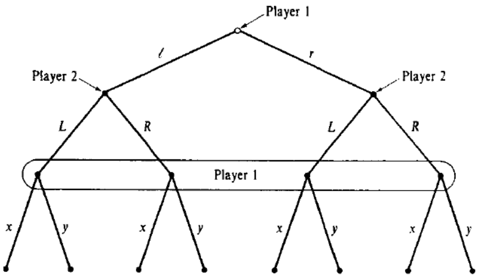
Remark: it is a basic postulate of game theory that all players know the structure of the game, know that their rivals know it, know that their rivals know that they know it, and so on – we say that the structure of the game is a common knowledge.

What do they know when they move?

Example of Violations



(a)



(b)

Game in figure (a) violates restriction 1, figure (b) violates restriction 2 (perfect recall).

What do they know when they move?

Imperfect vs. Incomplete Information

- If an agent cannot observe all her rival's previous moves, they are games of imperfect information.
- If an agent cannot observe all her rival's payoff, they are games of incomplete information.

We talk about games with incomplete information later.

What can they do?

Strategy

A central concept of game theory is the notion of a player's strategy. It gives a formal characterization of the player's "set of action".

A strategy is a *complete contingent plan*, or decision rule, that specifies how the player will act in *every possible distinguishable circumstance* in which she might be called upon to move.

Don't get confused between action (or whatever) and strategy!

Example: In Matching Pennies Version B, player 2's

- one action is playing H.
- one strategy is playing T if player 1 plays H, playing H if player 1 plays T.

When a player specifies her strategy, it is as if she had to write down an instruction book prior to the play so that the representative could act on her behalf merely by consulting that book.

What can they do?

Find Out All Possible (Pure) Strategies

Example 1: In Matching Pennies Version B, player 1 has two strategies:

- s1. play H.
- s2. play T.

Player 2 has four possible strategies:

- s1. play H if player 1 plays H; play H if player 1 plays T.
- s2. play H if player 1 plays H; play T if player 1 plays T.
- s3. play T if player 1 plays H; play H if player 1 plays T.
- s4. play T if player 1 plays H; play T if player 1 plays T.

Example 2: Find out strategies for players 1 and 2 in Matching Pennies Version B.

"Complete", "contingent" $\rightarrow$ a strategy needs to specify action in all possible situations!

A Side: Normal vs. Extensive Form of Presentation

1. We usually use extensive form to represent dynamic games, normal form to represent simultaneous games, but this is not a must.
2. With the help of a strategy, we can represent dynamic games with the normal form.

Player 2

		s_1	s_2	s_3	s_4
Player 1	H	−1, +1	−1, +1	+1, −1	+1, −1
	T	+1, −1	−1, +1	+1, −1	−1, +1

Figure 7.D.1

The normal form of Matching Pennies Version B.

A Side: Normal vs. Extensive Form of Presentation

3. With the help of the information set, we can represent simultaneous games with the extensive form.

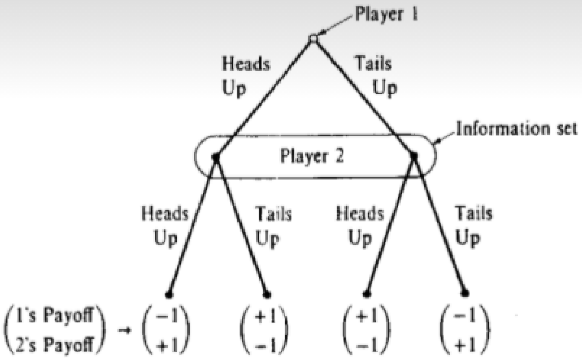


Figure 7.C.3
Extensive form for Matching Pennies Version C.

What can they do?

Randomized Actions: Mixed Strategy and Behavior Strategy

- When faced with a choice, people could respond with pure actions, or randomize.
- This gives pure strategy vs. 'random' strategy (or simply "mixed strategy" in most of the occasions).
- We have two *ways of present* an individual's 'random' strategy: mixed strategy and behavior strategy.

Mixed Strategy

Definition 8.1 (Mixed Strategy) Given player $i's$ (finite) pure strategy $s_i \in S_i$, a mixed strategy for player i,$\sigma_i : S_i \to [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Example:

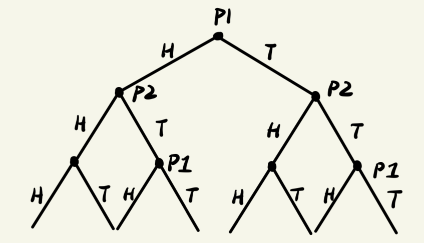
		Player 2	
		Head	Tail
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Figure: Normal form for Matching Pennies Version A

Behavior Strategy

Sometimes the set of pure strategies can be very large (especially in dynamic games), so for simplicity we let the player randomize separately over all possible actions at each of her information sets:

Example:



1. Player 1 has $2 \times 2 \times 2 = 8$ strategies
2. Player 1 has 5 decision nodes

Behavior Strategy

Definition 8.2 (Behavior Strategy) Given an extensive form game, a behavior strategy for player i specifies, for every information set $H \in \mathcal{H}$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\sum_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \mathcal{H}$.

Example:

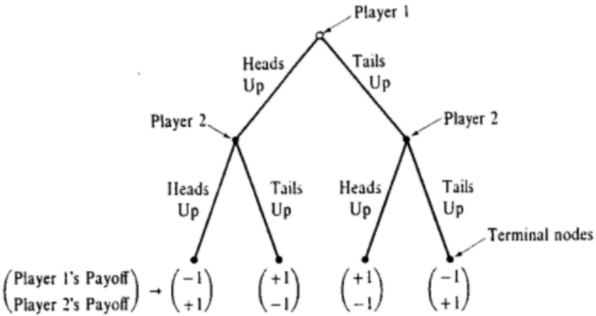


Figure 7.C.1
Extensive form for Matching Pennies Version B.

Notes:

1. The key difference between the definition of mixed strategy vs. behavior strategy is that the former randomizes over strategies (which are contingent on "irrelevant histories"), while the latter randomizes over information sets (which are not).

2. In MP version B, player 2 'randomizes' over 2 "nodes" using the definition of behavior strategy instead of 4 "nodes" using mixed strategy.

3. As might seem intuitive, for games of perfect recall (and we deal only with these), the two types of randomization are equivalent. So whichever form of randomized strategy we consider is therefore a matter of analytical convenience; we typically use behavior strategies when analyzing the extensive form representation of a game, and mixed strategies when analyzing the normal form.

Simultaneous-Move Games

Central question of game theory: what should we expect to observe in a game played by rational players who are fully knowledgeable about the structure of the game and each other's rationality?

1. What action can be 'rational' for players?
   - dominant and dominated strategies, iterated dominance
   - rationalize strategy
2. What equilibrium outcomes can arise from 'rational' for players?
   - Nash equilibrium
   - Bayesian Nash equilibrium (i.c.o. incomplete info)

Dominant and Dominated strategies

Definition 8.3-5.

Consider a strategy $s_i \in S_i$ for player i in a game $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ if for all $s'_i \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

We say:

- s_i is a strictly dominant strategy for player i.

- s'_i a strictly dominated for player i in game Γ_N.

- strategy s strictly dominates s'.

- If we change $>$ to $\geq$, we get *weak* dominance

- mixed strategies are permitted, we just need to change the notation of strategy from s_i to σ_i, the notation of strategy set from $\{S_i\}$ to $\{\Delta(S_i)\}$.

Iterated Deletion for Strictly Dominated Strategies

Example:

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	$-2, -2$	$-10, -1$
	Confess	$-1, -10$	$-5, -5$

Figure: The Prisoner's Dilemma

Iterated Deletion for Strictly Dominated Strategies

Design an Example Yourself:

Rationalized Strategies

A rational player will never choose strictly dominated strategies regardless of the strategy that her rivals will play.

We can eliminate more strategies by utilizing players' common knowledge of each other's rationality and the structure of the game.

Example:

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	$-\cancel{2}$ 3, -2	-10, -1
	Confess	-1, -10	-5, -5

Figure: An Variation of the Prisoner's Dilemma

In this example, DC is not strictly dominated by C for Prisoner 1; but given that he knows that if Prisoner 2 is rational she will never choose DC, then Prisoner 1 being a rational player will never choose DC either.

Definition 8.4 (Best response) In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$, strategy σ_i is a best response for player i to his rival's strategy σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \{\Delta(S_i)\}$. Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is the best response.

Definition 8.5 (Rationalizable strategies (RS))
In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$, the strategies in $\{\Delta(S_i)\}$ that survive the iterated removal of strategies that are never a best response are known as player i's rationalizable strategies.

Example:

Player 2

Player 1	b_1	b_2	b_3	b_4
a_1	0, 7	2, 5	7, 0	0, 1
a_2	5, 2	3, 3	5, 2	0, 1
a_3	7, 0	2, 5	0, 7	0, 1
a_4	0, 0	0, −2	0, 0	10, −1

Figure 8.C.1
$\{a_1, a_2, a_3\}$ are rationalizable strategies for player 1; $\{b_1, b_2, b_3\}$ are rationalizable strategies for player 2

Nash Equilibrium

Big picture

Nash equilibrium is the mostly used solution concept in the application of game theory to economics.

It is a solution concept that captures some degree of reality, but do not confuse it with reality.

There are also many other solution concepts (or equilibrium concepts) for strategic interactions – depending on what we believe would be a “reasonable” stable outcome when players interact strategically.

Nash Equilibrium

Definition

Definition 8.6 (Nash equilibrium) A strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ if for every player $i = 1, ..., I$,

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i})$$

for all $\sigma'_i \in \{\Delta(S_i)\}$.

In a Nash equilibrium, each player's strategy choice is a best response to the strategies actually played by his rivals. The blue words distinguish the concept of Nash equilibrium from that of rationalizability.

- RS is a necessary condition for NE.
- RS only requires that a player's strategy is a best response to some reasonable conjecture about what his rival will be playing; NE additionally requires players to be "correct" in their conjectures.

Example 1: Find out all pure-strategy NE of the following games:

		Player 2	
		Cinema	Football
Player 1	Cinema	$50, 100$	$0, 0$
	Football	$0, 0$	$100, 50$

		Player 2		
		l	m	r
Player 1	l	$5, 3$	$0, 4$	$3, 5$
	m	$4, 0$	$5, 5$	$4, 0$
	r	$3, 5$	$0, 4$	$5, 3$

Example 2: Find out all mixed-strategy NE of the following game:

		Player 2	
		Head	Tail
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Games of Incomplete Information: Bayesian Nash Equilibrium

We have assumed that players know all relevant information about each other, including the payoffs that each receives from various outcomes of the game. Such games are known as games of complete information.

However, do players really know their rivals' payoffs? Clearly, the answer is "no". In this case, players have what is known as incomplete information.

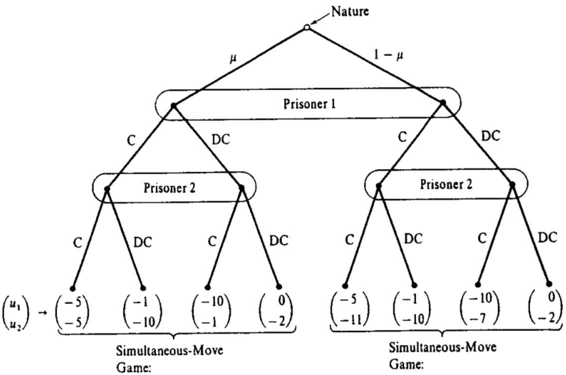
Fortunately, there is a widely-used approach (Harsanyi 1967-68) that can translate the game of incomplete information to a game of imperfect information.

In this approach, imagine each player's "type" is determined by a random variable ("Nature" rolls a dice).

The random variable's realization is observed only by the player, but its ex-ante probability is assumed to be a common knowledge among all players (i.e. player A knows B's type distribution, player B knows that A knows, and so on). (Big assumption, though not that crazy when we think a bit more)

Bayesian Game: Nature makes the first move, choosing *realizations* of the random variable that determines each player's type, and each player *only* observes the *realization* of his own random variable.

Example:



Prisoner 2

Prisoner 1	DC	C
DC	0, −2	−10, −1
C	−1, −10	−5, −5

Prisoner 2

Prisoner 1	DC	C
DC	0, −2	−10, −7
C	−1, −10	−5, −11

Figure 8.E.1
The DA's Brother game with incomplete information.

Above is a simultaneous game with incomplete information. We can rewrite it as a game tree with imperfect information. Prisoner 2 has two types, player 1 has one type (as indicated from the payoff function). Nature moves first, only prisoner 2 knows the realization of his type, i.e. which sub-tree they are in (as indicated by the information set).

We only consider pure strategies.

In this game, player 1 has two pure strategies: confess (C), don't confess (DC).

Player 2 has four pure strategies (think nature as a thrid player):

- C if type I, C if type II
- C if type I, DC if type II
- DC if type I, C if type II
- DC if type I, DC if type II

Bayesian Game

Formally, in a Bayesian game,

- each player i has a payoff function $u_i(s_i, s_{-i}, \theta_i)$, where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is only observed by player i.
- the joint probability distribution of the $\theta_i's$ is given by $F(\theta_1, ..., \theta_I)$, which is assumed to be common knowledge of all players.
- letting $\Theta = \Theta_1 \times ...\Theta_I$ summarize the "type space"

A **(pure strategy) Bayesian Game** is summarized by $[I, \{S_i\}, \{u_i(.)\}, \Theta, F()]$

Example: what is the corresponding $I, \{S_i\}, \{u_i(.)\}, \Theta, F(.)$ for the DA's brother game with incomplete information?

Bayesian Nash equilibrium

Player i's expected payoff given a profile of pure strategies for the I players is then given by:

$$\tilde{u}_i(s_i(.), ..., s_I(.)) = E_\theta[u_i(s_1(\theta_1), ..., s_I(\theta_I)), \theta_i].$$

A **(pure strategy) Bayesian Nash equilibrium** for the Bayesian game $[I, \{S_i\}, \{u_i(.)\}, \Theta, F()]$ is a profile of decision rules $(s_i(.), ..., s_I(.))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathscr{S}_i\}, \{\tilde{u}_i(.)\}]$. That is, for every $i = 1, 2, ..., I$,

$$\tilde{u}_i(s_i(.), s_{-i}(.)) \geq \tilde{u}_i(s_i'(.), s_{-i}(.))$$

for all $s_i'(.) \in \mathscr{S}_i$.

In other words, everyone's strategy should be mutually the best response conditional on his understanding of the probability distribution of his rival's types.

Example: How to solve for the (pure strategy) Bayesian Nash equilibrium of DA's brother game with incomplete information?

- Solve backwards:

1. type I player 2 will always choose C
2. type II player 2 will always choose DC
3. Given this behavior of player 2, player 1's best response is DC if $[-10u + 0(1-u)] > [-5u + 1(1-u)]$.
4. Therefore [DC, C if type I;DC if type II] is the BNE if $u \leq \frac{1}{6}$, [C, C if type I; DC if type II] is the BNE if $u \geq \frac{1}{6}$.