L3. Classical Demand Theory

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Literature

- MWG (1995), Chapter 3
- Kreps (1990), Chapter 2, Varian (1992), Chapters 7 - 8

Preferences and Utility Functions

We now apply the preference-based approach to demand theory in order to see what additional properties can be derived.

Definition 3.1 A preference relation $\succeq$ on X is **rational** if it is **complete** and **transitive**.

Notation

- $y \geq x$ means that $y_l \geq x_l$ for all $l \in \{1, \ldots, L\}$.
- $y > x$ means that $y_l \geq x_l$ for all $l \in \{1, \ldots, L\}$ and $y_k > x_k$ for at least one $k \in \{1, \ldots, L\}$.
- $y \gg x$ means that $y_l > x_l$ for all $l \in \{1, \ldots, L\}$.
- $\|y - x\| = \sqrt{\sum_{l=1}^{L} (y_l - x_l)^2}$

Definition 3.2 A preference relation $\succeq$ on X is

- **strongly monotone** if $x, y \in X$ and $y > x$ implies $y \succ x$.
- **monotone** if $x, y \in X$ and $y \gg x$ implies $y \succ x$.
- **locally non-satiated** if for every $x \in X$ and every $\epsilon > 0, \exists y \in X$ such that $\|y - x\| \leq \epsilon$ and $y \succ x$.

Interpretation and graphical illustration of monotonicity and non-satiation, dis-desirability.

Show that

- strong monotonicity implies monotonicity.
- monotonicity implies local non-satiation.
- non-satiation implies that indifference curves cannot be “thick”.

Definition 3.3 The preference relation $\succeq$ is **(strictly) convex** if for every $x \in X$ the upper contour set $\{y \in X \ y \succeq x\}$ is (strictly) convex; i.e., if $y \succeq x$ and $z \succeq x$, then $\alpha y + (1 - \alpha)z \succeq (\succ)x$ for any $\alpha \in (0, 1)$.

Interpretation and graphical illustration of convexity.

Definition 3.4 The preference relation $\succeq$ is **continuous** if for any sequence of pairs $\{x^n, y^n\}_{n=0}^{\infty}$ with $x^n \succeq y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succeq y$.

Remarks

- An alternative and equivalent definition of continuity requires that the upper and lower contour sets of $\succeq$ must be closed (contain their boundaries).
- Continuity implies that indifference curves cannot have jumps. Show this using the definition of continuity.
- Consider the lexicographic preference relation in the two goods case, i.e., $X = \mathbb{R}^2_+$ and $x \succeq y$ if and only if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Are these preferences continuous?

Proposition 3.1 Suppose that $X = \mathbb{R}^L_+$. If $\succeq$ is rational, monotone and continuous, then there is a utility function $u(x)$ that represents $\succeq$.

Proof sketch: We give a graphical sketch of the proof. The proof is constructive, i.e., it shows how a utility function that represents $\succeq$ can be constructed if $\succeq$ is rational, monotone and continuous.

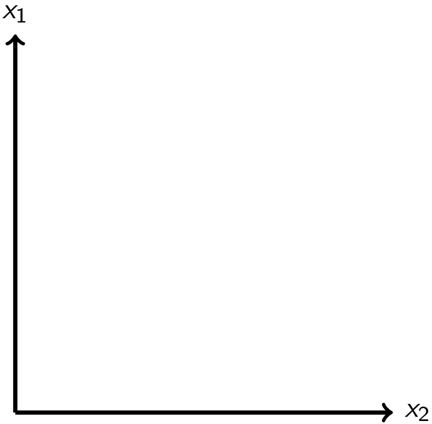

Figure: Figure 3.1: Construction of a utility function

Remarks

1. Recall that a utility function is unique only up to a positive, monotone transformation.
2. A much stronger proposition holds but is more difficult to prove: we do not have to require that $X = \mathbb{R}^L_t$ nor that $\succeq$ is monotone. Furthermore, it can be shown that there is a continuous utility function that represents $\succeq$. However, not all utility functions that represent $\succeq$ have to be continuous. Why not?
3. It is often convenient to work with a differentiable utility function. However, there are continuous preferences that cannot be represented by a differentiable utility function. Example?
   Thus, in order to guarantee the existence of a differentiable utility function we need an additional assumption on the smoothness of preferences.
4. If preferences are (strictly) convex, then any utility function that represents these preferences is (strictly) quasiconcave. This follows immediately from the definition of quasiconcavity.
5. However, convex preferences do not imply that the utility function is concave! Why not?

Utility Maximization and Demand Functions

In the following we will assume that the consumer's preferences satisfy the conditions of Proposition 3.1 and that his utility function is continuous and at least twice continuously differentiable.

Given the consumer's utility function his decision problem can now be expressed as a **utility maximization problem (UMP)**:

$$\max_{x \geq 0} u(x)$$

$$\text{s.t.} \quad p \cdot x \leq w$$

Proposition 3.2 If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proof:

Let $x(p, w)$ be the solution to UMP. This is called the Walrasian (sometimes also Marshallian) **demand correspondence**. If $x(p, w)$ is unique for all (p, w), then it is called the **demand function**.

Proposition 3.3 The demand correspondence $x(p, w)$ satisfies the following properties:

a) Homogeneity of degree 0 in (p, w)

b) Walras' Law

c) If $\succeq$ is strictly convex (so that $u(\cdot)$ is strictly quasiconcave), then $x(p, w)$ is unique for all (p, w).

Proof:

It can also be shown that $x(p, w)$ is continuous (upper hemicontinuous if $x(p, w)$ is a correspondence). Furthermore, if $x(p, w)$ is a demand function, then, under a mild additional assumption, it is also differentiable.

How to solve the UMP:

By Walras' Law, we know that the constraint $p \cdot x \leq w$ must hold with equality in the optimal solution. Thus, we can use Lagrange's method (with non-negativity constraints) to find the optimal consumption bundle.

Note

1. The Lagrange function is

$$\mathcal{L} = u(x) - \lambda[p \cdot x - w]$$

2. The Lagrange Theorem says that if $x^* \in x(p, w)$, then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $l \in \{1, \ldots, L\}$

$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{\partial u\left(x^*\right)}{\partial x_l} - \lambda p_l \leq 0$$

   with equality if $x_l^* > 0$. This condition is necessary but not sufficient for an optimal solution.

3. The UMP has a solution. The solution must satisfy Lagrange's condition. Hence, if there is only one (x^*, λ) satisfying Lagrange's condition, then it must be the solution.

4. Interpretation of λ?

5. Interpretation of

$$[\nabla u(x^*) - \lambda p] \cdot x^* = 0?$$

   Notation: $\nabla u(x^*) = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_L}\right)$

The Indirect Utility Function

Let $v(p, w) = u(x^*)$ for any $x^* \in x(p, w)$. Thus, $v(p, w)$ is the highest level of utility that the consumer can achieve given (p, w). This is called the **indirect utility function**.

Proposition 3.4 The indirect utility function $v(p, w) = u(x^*)$ satisfies the following properties:

(a) Homogeneity of degree 0 in (p, w).

(b) Strictly increasing in w.

(c) Nonincreasing in p_l for any l.

Proof:

Proposition 3.5 (Roy's Identity) Suppose that $v(p, w)$ is differentiable at (p, w). Then for every $l = 1, \ldots, L$:

$$x_l(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_l}}{\frac{\partial v(p,w)}{\partial w}}$$

Proof:

Remarks

1. The proof is an envelope theorem argument, but we did not use the envelope theorem explicitly.

2. To interprete the result, it is useful to write

$$\frac{\partial v(p,w)}{\partial p_l} = -x_l(p,w)\frac{\partial v(p,w)}{\partial w}$$

   Suppose that price p_l is reduced marginally by Δp_l. This gives additional "income" $\Delta p_l \cdot x_l(p,w)$ to the consumer. The marginal utility of a dollar is the same for all goods and equal to $\frac{\partial v(p,w)}{\partial w}$. Hence, the price reduction increases the consumer's utility by $\Delta p_l x_l(p,w)\frac{\partial v(p,w)}{\partial w}$.

3. Roy's identity is very useful. If we know the indirect utility function, then it is much easier to derive the demand function from the indirect utility than from the direct utility function.

The Dual Problem: Expenditure Minimization

Instead of maximizing the utility level for a given budget constraint, the consumer could also minimize his expenditures subject to the constraint that he achieves at least a given level of utility u:

$$\begin{aligned} &\min_{x} p \cdot x \\ \text{s.t.} \quad &u(x) \geq u \end{aligned}$$

The **expenditure minimization problem (EMP)** is the dual problem to the UMP: it reverses the roles of the objective function and the constraint.

However, the EMP also characterizes the efficient use of resources by the consumer. In fact, EMP and UMP are basically equivalent.

Proposition 3.6 Suppose that $p \gg 0$.

(a) If x^* is optimal in UMP when wealth is $w > 0$, then x^* is optimal in EMP when the required utility level is $u(x^*)$.

(b) If x^* is optimal in EMP when the required utility level is u, then x^* is optimal in UMP when wealth is $w = p \cdot x^*$.

Proof: We only prove (a).

The solution to EMP is denoted by $h(p, u)$ and is called the **Hicksian (or compensated) demand correspondence**, or function, if $h(p, u)$ is single-valued. Illustrate $h(p, u)$ graphically.

Proposition 3.7 For any $p \gg 0$ the Hicksian demand correspondence $h(p, u)$ has the following properties:

(a) Homogeneity of degree 0 in p.

(b) No excess utility: For any $x \in h(p, u), u(x) = u$.

(c) If $\succeq$ is strictly convex (so that $u(x)$ is strictly quasiconcave) then there is a unique solution to *EMP* for all (p, u).

Proof: Analogous to the proof of Proposition 3.3.

An important property of the Hicksian demand function is that it satisfies the **compensated law of demand**.

Proposition 3.8 Suppose that $h(p,u)$ is single valued for all $p \gg 0$. Then the Hicksian demand function satisfies the compensated law of demand, i.e., for all p' and p''

$$\left(p'' - p'\right) \cdot \left[h\left(p'', u\right) - h\left(p', u\right)\right] \leq 0.$$

Proof:

Let $e(p, u) = p \cdot x^*$, where $x^* \in h(p, u)$, be the **expenditure function** of the consumer.

Proposition 3.9 The expenditure function $e(p, u)$ is

(a) homogenous of degree one in p,

(b) strictly increasing in u,

(c) nondecreasing in p_l for any l,

(d) concave in p.

Intuition for these results? Explain (d) graphically.

Proof: (a) to (c) are straightforward and left as an exercise. To prove (d), consider two price vectors p' and p''. Let $\alpha \in [0,1]$ and $\bar{p} = \alpha p' + (1-\alpha)p''$. We have to show:

$$e(\bar{p}, u) \geq \alpha e\left(p', u\right) + (1-\alpha) e\left(p'', u\right)$$

Let x', x'' and $\bar{x}$ be the solutions to EMP at prices p', p'', and $\bar{p}$ respectively. It must be the case that:

$$\sum_{l=1}^{L} p_l' x_l' = e\left(p', u\right)$$

$$\sum_{l=1}^{L} p_l'' x_l'' = e\left(p'', u\right)$$

$$\sum_{l=1}^{L} \bar{p}_l \bar{x}_l = e(\bar{p}, u)$$

Since x' and x'' minimize expenditures at prices p' and p'', we have:

$$\sum_{l=1}^{L} p_l' \bar{x}_l \geq \sum_{l=1}^{L} p_l' x_l'$$

$$\sum_{l=1}^{L} p_l'' \bar{x}_l \geq \sum_{l=1}^{L} p_l'' x_l''$$

Multiplying the first inequality with α and the second one with $(1-\alpha)$ and adding up both inequalities yields

$$\sum_{l=1}^{L}\left[\alpha p_l'\bar{x}_l+(1-\alpha)p_l''\bar{x}_l\right]\geq\sum_{l=1}^{L}\alpha p_l'x_l'+\sum_{l=1}^{L}(1-\alpha)p_l''x_l''$$

Hence:

$$\begin{aligned}
e(\bar{p},u) &= \sum_{l=1}^{L}\bar{p}_l\bar{x}_l \\
&\geq \sum_{l=1}^{L}\alpha p_l'x_l'+\sum_{l=1}^{L}(1-\alpha)p_l''x_l'' \\
&= \alpha\cdot e\left(p',u\right)+(1-\alpha)e\left(p'',u\right)
\end{aligned}$$

Q.E.D.

The expenditure function can easily be derived from the Hicksian demand function by $e(p,u)=p\cdot h(p,u)$. However, the following result shows that we can also derive the Hicksian demand function from the expenditure function:

Remarks:

1. The proof is an envelope theorem argument, but we did not use the envelope theorem explicitly.
2. Interpretation: in finite approximation Shephard's Lemma says:

   $$\Delta e(p, u) = h_l(p, u) \cdot \Delta p_l$$

   If the price for good l increases by Δp_l, then the consumer has to pay $\Delta p_l h_l(p, u)$ in addition, in order to buy the same consumption bundle that he consumed before the price change. If Δp_l is very small, then this is also the amount necessary to get back to the old utility level u. The reason is that for small price changes and starting from an optimally chosen consumption bundle the substitution effects can be ignored.
3. Shephard's Lemma is very useful. If we know the expenditure function, it is much simpler to derive the Hicksian demand function via $e(p, u)$ than to derive it from the direct utility function.

Shephard's Lemma has several important implications that are summarized in the next proposition.

Proposition 3.10 (Shephard's Lemma) Suppose that $h(p, u) \gg 0$ is single valued and differentiable. Then

$$h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l}$$

for all $l = 1, \ldots, L$.

Proof:

Proposition 3.11 Suppose that $h(p, u)$ is single valued and continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(a) $D_p h(p, u) = D_p^2 e(p, u)$,

(b) $D_p h(p, u)$ is a negative semidefinite matrix,

(c) $D_p h(p, u)$ is a symmetric matrix,

(d) $D_p h(p, u) p = 0$.

Proof:

Remarks:

1. Negative semidefiniteness of $D_p h(p, u)$ is again the compensated law of demand. See Lecture 2! In particular, it implies that $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$, i.e., the own substitution effect is non-positive.
2. Symmetry of $D_p h(p, u)$ requires that

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial h_k(p, u)}{\partial p_l}$$

   This is not obvious and it is difficult to give an intuitive explanation for this unexpected result. We know already that symmetry of this matrix is not implied by the weak axiom. It is an additional property of the preference-based approach.
3. If $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$ then goods l and k are substitutes. If $\frac{\partial h_l(p,u)}{\partial p_k} < 0$, then they are called complements. Property (d) implies that for each good there exists at least one substitute. This follows immediately from $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$.

The Hicksian demand function is a very useful concept. It is particularly important when it comes to the evaluation of welfare changes. But, the Hicksian demand function is not directly observable. However, the following proposition shows that the Hicksian demand function $h(p, u)$ can be derived from the observable demand function $x(p, w)$.

Proposition 3.12 (Slutsky Equation) Suppose that $h(p, u)$ and $x(p, w)$ are single valued and differentiable. Then for all (p, w) and $u = v(p, w)$ we have

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} \cdot x_k(p, w).$$

Proof:

Remarks

1. The Slutsky Equation shows how the properties of the unobservable Hicksian demand function translate to the observable demand function.
2. In particular, the Slutsky Equation implies

$$\frac{\partial h_l(p,u)}{\partial p_l} = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} \cdot x_l(p,w)$$

3. If good l is a normal good ($\partial x_l / \partial w > 0$), then an increase in p_l reduces the Hicksian demand by less than the (Walrasian) demand. In the usual demand diagram (with p on the vertical axis), the Hicksian demand function is steeper than the (Walrasian) demand function.
4. If good l is an inferior good ($\partial x_l / \partial w < 0$), then an increase in p_l reduces the Hicksian demand by more than the (Walrasian) demand. In the usual demand diagram (with p on the vertical axis), the Hicksian demand function is less steep than the (Walrasian) demand function. A good can be a Giffen good only if it is inferior.
5. The Slutsky equation implies that the matrix of price derivatives of the Hicksian demand function, $D_p h(p,u)$, is equal to the Slutsky matrix $S(p,w)$ that we know already from Lecture 2. Note, however, that we derived the Slutsky matrix in Lecture 2 by using a different compensation rule for the consumer.

- In Lecture 2 (choice-based approach), we compensated the consumer for the price change by adjusting his wealth level so that he can still afford his old consumption bundle (Slutsky compensation).
- Here (preference-based approach) we compensate the consumer by adjusting his wealth so that he can still afford his old utility level (Hicks compensation).

Nevertheless, Proposition 3.12 shows that for small price changes the effect of both compensation rules is identical!

Putting it All Together

Because UMP and EMP are basically equivalent, we have:

1. $x_l(p, w) = h_l(p, v(p, w))$ The Walrasian demand at wealth w is equal to the Hicksian demand if the consumer wants to achieve at least the utility $v(p, w)$.
2. $h_l(p, u) = x_l(p, e(p, u))$ The Hicksian demand at utility level u is equal to the Walrasian demand if the consumer's wealth is equal to $e(p, u)$.
3. $v(p, e(p, u)) = u$ The indirect utility function is strictly increasing in w. Thus we can invert $v(p, \cdot)$, which is simply the expenditure function.
4. $e(p, v(p, w)) = w$ The expenditure function is strictly increasing in u. Thus we can invert $e(p, \cdot)$, which is simply the indirect utility function.

Let us now summarize the relationship between UMP and EMP, between Hicksian and Walrasian demand and between indirect utility and expenditure function in the following diagram:

We have shown that the preference based approach to consumption theory has the following implications for the (Walrasian) demand function $x(p, w)$:

1. **Homogeneity of degree zero**
2. **Walras' law**
3. **Compensated Law of Demand (Slutsky matrix is negative semi-definite)**
4. **Symmetry of the Slutsky matrix**

Remarks

1. A natural question is whether there are any other properties of the demand function that are implied by the preference-based approach. The answer is no. It can be shown that for any demand function that satisfies (1) to (4) there exists a utility function (representing a rational preference relation) such that the demand function is generated by this utility function. This is known as the Integrability Problem (see MWH, Chapter 3.*H*). It also shows, how the preferences of the consumer can (almost, but not quite) be recovered from his observed demand behavior.

2. Properties (1) to (3) are also implied by the Weak Axiom of the choice-based approach. Hence, symmetry of the Slutsky matrix is the only additional property that the preference based approach gives us.

3. One could ask whether it is possible to impose an additional assumption in the choice-based approach that also yields a symmetric Slutsky matrix. This assumption is the **Strong Axiom of Revealed Preference**.

Definition 3.5 The demand function $x(p, w)$ satisfies the **Strong Axiom of Revealed Preference** if for any list

$$\left(p^1, w^1\right), \ldots, \left(p^N, w^N\right)$$

with $x\left(p^{n+1}, w^{n+1}\right) \neq x\left(p^n, w^n\right)$ for all $n \leq N-1$, we have
$p^N \cdot x\left(p^1, w^1\right) > w^N$ whenever $p^n \cdot x\left(p^{n+1}, w^{n+1}\right) \leq w^n$ for all $n \leq N-1$.
This definition says that if $x\left(p^1, w^1\right)$ is directly or indirectly (through the chain of $x\left(p^n, w^n\right)$) revealed preferred to $x\left(p^N, w^N\right)$, then $x\left(p^N, w^N\right)$ cannot be revealed preferred to $x\left(p^1, w^1\right)$, because $x\left(p^1, w^1\right)$ is not affordable at $\left(p^N, w^N\right)$.

The Strong Axiom is essentially equivalent to the preference-based approach presented here.

4. While most positive results of consumption theory can also be derived from the choice-based approach, the normative evaluation of welfare changes requires the preference-based approach.