Technical Note

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Monopolistic Competition

1. Monopolistic competition

Consider a country H where each firm has monopoly power over a single variety x_j. A firm pays a fixed cost f and a variable cost b, so it hires labor according to

$$l_j = f + bx_j$$

Suppose the representative consumer has L_H units of labor for which he receives a wage w. The consumer has utility over N differentiated goods given by

$$U = \left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

a. Show that demand for good j is given by

$$q_j = \frac{p_j^{-\sigma}}{\sum_{k=1}^{N} p_k^{1-\sigma}} wL_H$$

Answer:

The utility maximization problem of the consumer is given by:

$$\max_{q_j} U = \left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$s.t. \quad \sum_{j=1}^{N} q_j p_j = wL_H \equiv E$$

To solve the constrained optimization problem we write down the Lagrangian:

$$\mathcal{L} = \left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} + \lambda(E - \sum_{j=1}^{N} q_j p_j).$$

where λ is the lagrangier multiplier.

Take the F.O.C w.r.t. q_j:

$$\left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}-1} q_j^{\frac{\sigma-1}{\sigma}-1} = \lambda p_j, \tag{1}$$

Multiply both side by q_j, and sum over $j = 1, 2, ..., N$:

$$\left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}-1} \sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}} = \lambda \sum_{j=1}^{N} p_j q_j,$$

$$\rightarrow \qquad U \equiv \left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} = \lambda E,$$

which implies that

$$1/\lambda = E/U,$$

i.e., $1/\lambda$ reflects the shadow price of the consumption bundle, $\left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$. Hence we denote that

$$1/\lambda \equiv P.$$

Plug $1/\lambda \equiv P.$ back into (2):

$$\left[\sum_{j=1}^{N} q_j^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}-1} q_j^{\frac{\sigma-1}{\sigma}-1} = p_j/P. \tag{2}$$

$$Q^{\frac{1}{\sigma}} q_j^{-\frac{1}{\sigma}} = p_j/P$$

$$\rightarrow \qquad q_j = (\frac{p_j}{P})^{-\sigma} Q$$

$$\rightarrow \qquad q_j = \frac{p_j^{-\sigma}}{P^{1-\sigma}} PQ = \frac{p_j^{-\sigma}}{P^{1-\sigma}} E = \frac{p_j^{-\sigma}}{P^{1-\sigma}} wL_H.$$

Now we left showing that

$$P^{1-\sigma} = \sum_{k=1}^{N} p_k^{1-\sigma}.$$

Multiply both sides of $q_j = \frac{p_j^{-\sigma}}{P^{1-\sigma}} wL_H$ *by* p_j*:*

$$p_j q_j = \frac{p_j^{1-\sigma}}{P^{1-\sigma}} wL_H.$$

Then sum over $j = 1, ..., N$ *over both sides:*

$$\sum_{j=1}^{N} p_j q_j = \sum_{j=1}^{N} \frac{p_j^{1-\sigma}}{P^{1-\sigma}} wL_H.$$

As $\sum_{j=1}^{N} p_j q_j = wL_H$*, above equation simplifies to:*

$$1 = \sum_{j=1}^{N} \frac{p_j^{1-\sigma}}{P^{1-\sigma}}.$$

Note P *does not depends on* $\sum_{j=1}^{N}$*. Take it out and arrange terms:*

$$P^{1-\sigma} = \sum_{j=1}^{N} p_j^{1-\sigma}.$$

Then change the subscript from j *to* k *(just a notation change to avoid duplicacy)*

- Q.D.E.

b. What is the optimal price for each variety?

Answer:

Each firm chooese the optimal price to maximize its profit. A firm's optimization *problem is given by:*

$$\max_{p_j} \pi_j = p_j q_j - b q_j w - w f \tag{3}$$

$$s.t. \quad q_j = \frac{p_j^{-\sigma}}{P^{1-\sigma}} w L_H.$$

Note that under monopolistic competition, firm behaves as if their behavior has no impact on aggregate economic variables, such as P and w. F.O.C. w.r.t. p_j

$$q_j + p_j \frac{\partial q_j}{\partial p_j} - bw \frac{\partial q_j}{\partial p_j} = 0$$

$$\rightarrow \qquad p_j - bw = -q_j (\frac{\partial q_j}{\partial p_j})^{-1}$$

Using the demand constraint $\quad q_j = \frac{p_j^{-\sigma}}{P^{1-\sigma}} w L_H$, *one can show that* $-q_j(\frac{\partial q_j}{\partial p_j})^{-1} = \frac{p_j}{\sigma}$. *Therefore:*

$$p_j - bw = \frac{p_j}{\sigma} \qquad \rightarrow \qquad p_j = \frac{\sigma b w}{\sigma - 1}.$$

c. Compute the equilibrium number of varieties.

Answer:

Because of free entry, each firm earns zero profit:

$$p_j q_j - b q_j w - w f = 0$$

Given a firm charges the optimal price $p_j = \frac{\sigma b w}{\sigma - 1}$,

$$\frac{\sigma b w}{\sigma - 1} q_j - b q_j w - w f = 0 \quad \rightarrow \quad q_j = \frac{(\sigma - 1) f}{b}.$$

Given the total production, the firm's total employment equals:

$$l_j = f + b q_j = \sigma f.$$

Note that all firms are 'identical (or, symmetric)' in this model. Hence total employment should be the same across firms. Hence the labor market clearing condition implies:

$$l_j N = L_H \quad \rightarrow \quad N = \frac{L_H}{\sigma f}.$$